A CERTAIN TWISTED JACQUET MODULE OF GL(4) OVER A FINITE FIELD

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Abstract. Let F be a finite field and $G = \mathrm{GL}(4, F)$. In this paper, we explicitly calculate a certain twisted Jacquet module of an irreducible cuspidal representation of G.

1. Introduction

Let F be a finite field and $G = \mathrm{GL}(n, F)$. Let P be a parabolic subgroup of G with Levi decomposition $P = MN$. Let π be any irreducible finite dimensional complex representation of G and ψ be an irreducible representation of N. Let $\pi_{N,\psi}$ be the sum of all irreducible representations of N inside π, on which π acts via the character ψ. It is easy to see that $\pi_{N,\psi}$ is a representation of the subgroup M_ψ of M, consisting of those elements in M which leave the isomorphism class of ψ invariant under the inner conjugation action of M on N. The space $\pi_{N,\psi}$ is called the *twisted Jacquet module* of the representation π. It is an interesting question to understand for which irreducible representations π, we have $\pi_{N,\psi}$ is non-zero and to understand the structure of $\pi_{N,\psi}$ as a module for M_ψ. When P is the Borel subgroup of G and ψ is a non-degenerate character of N, a well known result of Gelfand and Graev [2] says that $\pi_{N,\psi}$ is at most one dimensional. There is also the work of Kawanaka [6] on generalized Gelfand-Graev representations.

In this paper, motivated by the work of Prasad in [7], we study the structure of a certain twisted Jacquet Module of a cuspidal representation of $\mathrm{GL}(4, F)$. Before we state our result, we set up some notation and mention the work of Prasad. Let $G = \mathrm{GL}(2n, F)$ and $P = MN$ be the standard maximal parabolic subgroup of G corresponding to the partition (n, n). Then, $M \simeq \mathrm{GL}(n, F) \times \mathrm{GL}(n, F)$ and $N \simeq M(n, F)$. Let ψ be any character of $N \simeq \mathrm{M}(n, F)$ and ψ_0 be a fixed non-trivial character of F. It is easy to see that there exists $A \in \mathrm{M}(n, F)$ such that $\psi = \psi_A$, where $\psi_A(X) = \psi_0(\mathrm{Tr}(AX))$. The group $\mathrm{GL}(n, F) \times \mathrm{GL}(n, F)$ acts on the set of characters of $\mathrm{M}(n, F)$ via,

$$(g_1, g_2).\psi_A = \psi_{g_2^{-1} A g_1}$$

and we get a decomposition of the set of characters of $\mathrm{M}(n, F)$ into disjoint orbits with respect to the above action. For $0 \leq i \leq n$, we let

$$A_i = \begin{bmatrix} I_i & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{M}(n, F),$$

where I_i is the identity matrix in $\mathrm{GL}(i, F)$. The matrices $A_i, 0 \leq i \leq n$ form a set of representatives for the orbits under the above action. When $i = n$, the character ψ_{A_n} is a representative for the orbit of the non-degenerate characters of $\mathrm{M}(n, F)$.

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In [7], Prasad explicitly describes π_{N,ψ_A} as a module for M_{ψ_A}, when π is an irreducible cuspidal representation of $\mathrm{GL}(2n, F)$, ψ_A is the character of N given by $\psi_A(X) = \psi_0(\mathrm{Tr}(AX))$ where $A = A_n$. More recently in [4], the work of Prasad was generalized to $\mathrm{GL}(kn)$.

In this paper, we explicitly calculate the twisted Jacquet module of an irreducible cuspidal representation π of $\mathrm{GL}(4, F)$ when $A = A_1 \in \mathrm{M}(2, F)$. In other words, we take ψ_A to be a degenerate character of $\mathrm{M}(2, F)$ and calculate the twisted Jacquet module π_{N,ψ_A}. Before we state our theorem, we set up some notation. We write F_m for the unique field extension of F of degree m. Let

$$L = \left\{ \begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix} \mid y \in F^\times, x \in F \right\}, U = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in F \right\} \text{ and}$$

$$Z = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in F^\times \right\} \simeq F^\times.$$

We write $\overline{L}$ and $\overline{U}$ for the opposite of L and U respectively. We write μ for a fixed non-trivial additive character of U and $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We write $\mu^w : \overline{U} \to \mathbb{C}^\times$ for the character of $\overline{U}$ given by

$$\mu^w \left(\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \right) = \mu \left(w \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} w^{-1} \right) = \mu \left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right).$$

Theorem 1.1. *Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of $\mathrm{GL}(4, F)$ attached to a regular character θ of $F_4^\times$. Let ψ_0 be a non-trivial additive character of F and $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Let ψ_A be the character of N given by $\psi_A(X) = \psi_0(\mathrm{Tr}(AX))$. Then,*

$$\pi_{N,\psi_A} \simeq (\theta|_{F^\times} \otimes \mathrm{ind}_{\bar{U}}^{\bar{L}} \mu^w) \otimes \mathrm{ind}_U^L \mu$$

as M_{ψ_A} modules.

We establish the above isomorphism by explicitly calculating the characters of π_{N,ψ_A} and $(\theta|_{F^\times} \otimes \mathrm{ind}_{\bar{U}}^{\bar{L}} \mu^w) \otimes \mathrm{ind}_U^L \mu$, and showing that they are equal at any arbitrary element of M_{ψ_A}. Currently we are investigating the problem for $\mathrm{GL}(2n, F)$. We will write up the details at a later time.

It is also interesting to study the case when the finite field is replaced with a p-adic field. Our hope is that understanding the problem for the finite group case might help in understanding the problem in the p-adic case. We hope to study these problems in future.

2. Preliminaries

In this section, we mention some preliminary results that we need in our paper.

2.1. **Character of a Cuspidal Representation.**

Let F be the finite field of order q and $G = \mathrm{GL}(m, F)$. The representation theory of $\mathrm{GL}(m, F)$ is due to J.A. Green [5]. In this section, we recall some results about computing the character values of a cuspidal representation. Let F_m be the unique field extension of F of degree m. A character θ of $F_m^\times$ is called a "regular" character, if under the action of the Galois group of F_m over F, θ gives rise to m distinct characters of $F_m^\times$. It is a well known fact that the cuspidal representations of $\mathrm{GL}(m, F)$ are parametrized by the regular characters of $F_m^\times$. To avoid introducing more notation, we mention below only the relevant statements on computing the character values that we have used.

Theorem 2.1 (Green)**.** *Let θ be a regular character of $F_m^\times$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of* $\mathrm{GL}(m, F)$ *associated to θ. Let Θ_θ be its character. If $g \in \mathrm{GL}(m, F)$ is such that the characteristic polynomial of g is not a power of a polynomial irreducible over F. Then, we have*

$$\Theta_\theta(g) = 0.$$

See Page 130 in [3] for the statement of the above theorem.

Theorem 2.2 (Green)**.** *Let θ be a regular character of $F_m^\times$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of* $\mathrm{GL}(m, F)$ *associated to θ. Let Θ_θ be its character. Suppose that $g = s.u$ is the Jordan decomposition of an element g in* $\mathrm{GL}(m, F)$*. If $\Theta_\theta(g) \neq 0$, then the semisimple element s must come from $F_m^\times$. Suppose that s comes from $F_m^\times$. Let z be an eigenvalue of s in F_m and let t be the dimension of the kernel of $g - z$ over F_m. Then*

$$\Theta_\theta(g) = (-1)^{m-1}\left[\sum_{\alpha=0}^{d-1} \theta(z^{q^\alpha})\right](1 - q^d)(1 - (q^d)^2)\dots(1 - (q^d)^{t-1})$$

where q^d is the cardinality of the field generated by z over F, and the summation is over the distinct Galois conjugates of z.

See Theorem 2 in [7] for this version.

2.2. **Twisted Jacquet Module.** In this section, we recall the character and the dimension formula of the twisted Jacquet module of a representation π.

Let $G = \mathrm{GL}(k, F)$ and $P = MN$ be a parabolic subgroup of G. Let ψ be a character of N. For $m \in M$, let ψ^m be the character of N defined by $\psi^m(n) = \psi(mnm^{-1})$. Let

$$V(N, \psi) = \mathrm{Span}_{\mathbb{C}}\{\pi(n)v - \psi(n)v \mid n \in N, v \in V\}$$

and

$$M_\psi = \{m \in M \mid \psi^m(n) = \psi(n), \forall n \in N\}.$$

Clearly, M_ψ is a subgroup of M and it is easy to see that $V(N, \psi)$ is an M_ψ-invariant subspace of V. Hence, we get a representation $(\pi_{N,\psi}, V/V(N, \psi))$ of M_ψ. We call $(\pi_{N,\psi}, V/V(N, \psi))$ the twisted Jacquet module of π with respect to ψ. We write $\Theta_{N,\psi}$ for the character of $\pi_{N,\psi}$.

Proposition 2.3. *Let (π, V) be a representation of* $\mathrm{GL}(k, F)$ *and Θ_π be the character of π. We have*

$$\Theta_{N,\psi}(m) = \frac{1}{|N|}\sum_{n \in N} \Theta_\pi(mn)\overline{\psi(n)}.$$

Proof. Consider the projection of V given by

$$P_{N,\psi}(v) = \frac{1}{|N|}\sum_{n \in N} \pi(n)\psi^{-1}(n)v.$$

We have,

$$\begin{aligned}\Theta_{N,\psi}(m) &= \mathrm{Tr}(\pi_{N,\psi}(m)) \\ &= \mathrm{Tr}(\pi(m)|_{V/V(N,\psi)}) \\ &= \mathrm{Tr}(\pi(m)\circ P_{N,\psi}) \\ &= \frac{1}{|N|}\sum_{n\in N}\mathrm{Tr}(\pi(mn))\overline{\psi(n)} \\ &= \frac{1}{|N|}\sum_{n\in N}\Theta_\pi(mn)\overline{\psi(n)}.\end{aligned}$$

□

Remark 2.4. Taking $m=1$, we get the dimension of $\pi_{N,\psi}$. To be precise, we have

$$\dim_{\mathbb{C}}(\pi_{N,\psi}) = \frac{1}{|N|}\sum_{n\in N}\Theta_\pi(n)\overline{\psi(n)}.$$

2.3. **Character of the induced representation.** In this section, we recall the character formula for the induced representation of a group G. For a proof, we refer the reader to Chapter 3, Theorem 12 in [8].

Proposition 2.5. *Let G be a finite group and H be a subgroup of G. Let (π, V) be a representation of H and χ_π be the character of π. Then for each $s\in G$, the character of $\mathrm{ind}_H^G(\pi)$ is given by*

$$\chi_{\mathrm{ind}_H^G(\pi)}(s) = \frac{1}{|H|}\sum_{\substack{t\in G\\ tst^{-1}\in H}}\chi_\pi(tst^{-1}).$$

3. Preliminary calculations for computing the dimension

In this section, we make some preliminary calculations that we need to compute the dimension of the twisted Jacquet module that we study in this paper.

Let $\mathrm{M}(n,m,r,q)$ denote the set of $n\times m$ matrices of rank r over the finite field F of cardinality q. It is well known that we have

$$\#\,\mathrm{M}(n,m,r,q) = \prod_{j=0}^{r-1}\frac{(q^n-q^j)(q^m-q^j)}{(q^r-q^j)}.$$

For an elementary proof of this fact, we refer the reader to Theorem 2 in [1].

In our case, we have $n=m=2$ and $r\in\{0,1,2\}$. Using the above formula we can compute the number of matrices in $\mathrm{M}(2,F)$ with a fixed rank. We summarize this information in the following table.

Table 1. Number of matrices in $\mathrm{M}(2,F)$ with a fixed rank r

r	0	1	2
$\#\,\mathrm{M}(2,2,r,q)$	1	$(q^2-1)(q+1)$	$(q^2-1)(q^2-q)$

Let $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$. For $\alpha\in F$, consider the subset $Y_{2,r}^\alpha$ of $\mathrm{M}(2,F)$ given by

$$Y_{2,r}^\alpha = \{X\in\mathrm{M}(2,F)\mid \mathrm{Rank}(X)=r, \mathrm{Tr}(AX)=\alpha\}. \tag{3.1}$$

We compute the cardinality of the subset $Y_{2,r}^{\alpha}$ for a fixed rank $r \in \{0,1,2\}$ and a fixed trace $\alpha \in F$. We give the details of the calculation below.

For $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $AX = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$. Suppose that $\mathrm{Rank}(X) = r \in \{0,1,2\}$ and $\mathrm{Tr}(AX) = \alpha \in F$. Let $Y_{2,r}^{\alpha}$ be as in (3.1).

Lemma 3.1. *Let $\alpha, \beta \in F^{\times}$. Then we have*

$$\#Y_{2,r}^{\alpha} = \#Y_{2,r}^{\beta}.$$

Proof. Let $X \in Y_{2,r}^{\alpha}$. Since $\mathrm{Tr}(AX) = \alpha$, we have $X = \begin{bmatrix} \alpha & b \\ c & d \end{bmatrix}$. Let $K = \begin{bmatrix} \beta\alpha^{-1} & 0 \\ 0 & 1 \end{bmatrix}$. Consider the map $\phi : Y_{2,r}^{\alpha} \to Y_{2,r}^{\beta}$ given by

$$\phi(X) = KX.$$

Since K is invertible, it follows that ϕ is injective. For $Z = \begin{bmatrix} \beta & p \\ n & s \end{bmatrix} \in Y_{2,r}^{\beta}$, let $X = \begin{bmatrix} \alpha & p\alpha\beta^{-1} \\ n & s \end{bmatrix}$. Then $X \in Y_{2,r}^{\alpha}$ and $\phi(X) = Z$. Hence the result. $\square$

From Lemma 3.1, it is enough to consider the following cases to count the cardinality of $Y_{2,r}^{\alpha}$.

a) $r = 0$, $\alpha \in F$.
b) $r = 1$, $\alpha = 0$.
c) $r = 1$, $\alpha = 1$.
d) $r = 2$, $\alpha = 0$.
e) $r = 2$, $\alpha = 1$.

In case a), since $r = 0$, it follows that X is the zero matrix. Hence it follows that

$$\#Y_{2,0}^{\alpha} = \begin{cases} 1, & \alpha = 0 \\ 0, & \alpha \neq 0. \end{cases}$$

In case b), since $\alpha = 0$, we have $a = 0$. Therefore,

$$X = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}.$$

If $c = 0$, since $r = 1$, $X_2 \neq 0$. Therefore, we get $(q^2 - 1)$ such matrices. If $c \neq 0$, since $r = 1$, we have $X_2 = \beta X_1$ for some $\beta \in F$. It follows that

$$X = \begin{bmatrix} 0 & 0 \\ c & \beta c \end{bmatrix}.$$

Therefore, we get $(q-1)q$ such matrices. Thus we have

$$\#Y_{2,1}^{0} = 2q^2 - q - 1.$$

In case c), since $\alpha = 1$, we have $a = 1$. Therefore,

$$X = \begin{bmatrix} 1 & b \\ c & d \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}.$$

Since $r = 1$, if $c = 0$, we have $X_1 \neq 0$, $X_2 = \beta X_1$. Hence

$$X = \begin{bmatrix} 1 & \beta \\ 0 & 0 \end{bmatrix}.$$

Therefore, we get q such matrices. If $c \neq 0$, since $r = 1$, we have $X_2 = \beta X_1$ for some $\beta \in F$. It follows that

$$X = \begin{bmatrix} 1 & \beta \\ c & \beta c \end{bmatrix}.$$

Therefore, we get $(q-1)q$ such matrices. Thus we have

$$\#Y^1_{2,1} = q^2.$$

In case d), since $\alpha = 0$, we have $a = 0$. Therefore,

$$X = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}.$$

Since $r = 2$, we have $c \neq 0$, and $X_2 \neq \beta X_1$. We can choose X_1 in $(q-1)$ ways and X_2 in $(q^2 - q)$ ways. Therefore, we get $(q-1)(q^2-q)$ such matrices. Thus we have

$$\#Y^0_{2,2} = q(q-1)^2.$$

In case e), since $\alpha = 1$, we have $a = 1$. Therefore,

$$X = \begin{bmatrix} 1 & b \\ c & d \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}.$$

Since $r = 2$, if $c = 0$, we have

$$X = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$$

where $X_1 \neq 0$, $X_2 \neq \beta X_1$. Thus we have $(q^2 - q)$ such matrices. If $c \neq 0$, since $r = 2$, we have

$$X = \begin{bmatrix} 1 & b \\ c & d \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$$

where $X_1 \neq 0$, $X_2 \neq \beta X_1$. We can choose X_1 in $(q-1)$ ways and X_2 in $(q^2 - q)$ ways. Therefore, we get $(q-1)(q^2-q)$ such matrices. Thus we have

$$\#Y^1_{2,2} = q^2(q-1).$$

We summarize the details of the above computations in the following table.

Table 2. Cardinality of $Y^{\alpha}_{2,r}$

r	0	1	2
$\alpha = 0$	1	$2q^2 - q - 1$	$q(q-1)^2$
$\alpha \neq 0$	0	q^2	$q^2(q-1)$

4. Dimension of the twisted Jacquet module

Let θ be a regular character of $F_4^{\times}$ and let $\pi = \pi_\theta$ be an irreducible cuspidal representation of $\mathrm{GL}(4, F)$. We write Θ_θ for the character of $\pi = \pi_\theta$. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\psi_A : N \to \mathbb{C}^{\times}$ be the character of N given by

$$\psi_A \left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \right) = \psi_0(\mathrm{Tr}(AX)). \tag{4.1}$$

In this section, we calculate the dimension of π_{N,ψ_A}. Before we continue, we record a preliminary lemma that we need.

Lemma 4.1. *Let $r \in \{0,1,2\}$ and $X \in \mathrm{M}(2,2,r,q)$. We have*

$$\Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right) = \begin{cases} (q-1)(q^2-1)(q^3-1), & \text{if } r=0\\ -(q-1)(q^2-1), & \text{if } r=1\\ (q-1), & \text{if } r=2. \end{cases}$$

Proof. The character values can be computed using Theorem 2.2 above. $\square$

Theorem 4.2. *Let θ be a regular character of $F_4^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of* $\mathrm{GL}(4,F)$*. We have*

$$\dim_{\mathbb{C}}(\pi_{N,\psi_A}) = (q-1)^2.$$

Proof. It is easy to see that the dimension of π_{N,ψ_A} is given by

$$\dim_{\mathbb{C}}(\pi_{N,\psi_A}) = \frac{1}{q^4}\sum_{X\in\mathrm{M}(2,F)} \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\overline{\psi_0(\mathrm{Tr}(AX))}. \tag{4.2}$$

We calculate the following sums

a) $S_1 = \sum_{X\in\mathrm{M}(2,2,0,q)} \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\overline{\psi_0(\mathrm{Tr}(AX))}$

b) $S_2 = \sum_{\substack{X\in\mathrm{M}(2,2,1,q)\\ \mathrm{Tr}(AX)=0}} \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\overline{\psi_0(\mathrm{Tr}(AX))} + \sum_{\substack{X\in\mathrm{M}(2,2,1,q)\\ \mathrm{Tr}(AX)=\alpha\neq 0}} \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\overline{\psi_0(\mathrm{Tr}(AX))}$

c) $S_3 = \sum_{\substack{X\in\mathrm{M}(2,2,2,q)\\ \mathrm{Tr}(AX)=0}} \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\overline{\psi_0(\mathrm{Tr}(AX))} + \sum_{\substack{X\in\mathrm{M}(2,2,2,q)\\ \mathrm{Tr}(AX)=\alpha\neq 0}} \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\overline{\psi_0(\mathrm{Tr}(AX))}$

separately to compute the dimension of π_{N,ψ_A}.

For a), we clearly have

$$\begin{aligned} S_1 &= \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\overline{\psi_0(0)}\\ &= (q-1)(q^2-1)(q^3-1). \end{aligned}$$

For b), we have

$$S_2 = \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\left(\sum_{\substack{X\in\mathrm{M}(2,2,1,q)\\ \mathrm{Tr}(AX)=0}}\overline{\psi_0(0)} + \sum_{\substack{X\in\mathrm{M}(2,2,1,q)\\ \mathrm{Tr}(AX)=\alpha\neq 0}}\overline{\psi_0(\alpha)}\right)$$

$$= \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)(\#Y_{2,1}^0 - \#Y_{2,1}^1)$$

$$= -(q-1)(q^2-1)(q^2-q-1).$$

For c), we have

$$S_3 = \Theta_\theta\left(\begin{bmatrix}1 & X\\0 & 1\end{bmatrix}\right)\left(\sum_{\substack{X\in \mathrm{M}(2,2,2,q)\\ \mathrm{Tr}(AX)=0}} \overline{\psi_0(0)} \quad + \sum_{\substack{X\in \mathrm{M}(2,2,2,q)\\ \mathrm{Tr}(AX)=\alpha\neq 0}} \overline{\psi_0(\alpha)}\right)$$

$$= \Theta_\theta\left(\begin{bmatrix}1 & X\\0 & 1\end{bmatrix}\right)(\#Y^0_{2,2} - \#Y^1_{2,2})$$

$$= (q-1)(q-q^2).$$

From (4.2), it follows that

$$\begin{aligned}\dim_{\mathbb{C}}(\pi_{N,\psi_A}) &= \frac{1}{q^4}\{S_1+S_2+S_3\}\\ &= \frac{1}{q^4}\{(q-1)(q^2-1)(q^3-1)-(q-1)(q^2-1)(q^2-q-1)+(q-1)(q-q^2)\}\\ &= (q-1)^2.\end{aligned}$$

□

5. Main theorem

Let $G = \mathrm{GL}(4,F)$ and P be the maximal parabolic subgroup of G. We have $P = MN$, where $M \simeq \mathrm{GL}(2,F)\times \mathrm{GL}(2,F)$ and $N \simeq \mathrm{M}(2,F)$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of G where θ is a regular character of $F_4^\times$. Let ψ_0 be a fixed non-trivial additive character of F. In this section, we explicitly calculate the twisted Jacquet module π_{N,ψ_A} where ψ_A is the character of N defined in (4.1). Before we state our main result, we recall some notation and record a few preliminary lemmas that we need.

Let $L = \left\{\begin{bmatrix}1 & x\\0 & y\end{bmatrix} \mid y\in F^\times, x\in F\right\}$, $U = \left\{\begin{bmatrix}1 & x\\0 & 1\end{bmatrix} \mid x\in F\right\}$ and $Z = \left\{\begin{bmatrix}a & 0\\0 & a\end{bmatrix} \mid a \in F^\times\right\}$. We write $\overline{L}$ and $\overline{U}$ for the opposite of L and U respectively. We write μ for a fixed non-trivial additive character of U and $w = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$. We write $\mu^w : \overline{U}\to\mathbb{C}^\times$ for the character of $\overline{U}$ given by

$$\mu^w\left(\begin{bmatrix}1 & 0\\x & 1\end{bmatrix}\right) = \mu\left(w\begin{bmatrix}1 & 0\\x & 1\end{bmatrix}w^{-1}\right) = \mu\left(\begin{bmatrix}1 & x\\0 & 1\end{bmatrix}\right).$$

Lemma 5.1. *Let $M_{\psi_A} = \{m\in M \mid \psi_A^m(n) = \psi_A(n), \forall n\in N\}$. Then we have*

$$M_{\psi_A} = \left\{\begin{bmatrix}a & 0 & & \\ c & d & & \\ & & a & r\\ & & 0 & s\end{bmatrix} \mid a,d,s\in F^\times, c,r\in F\right\}.$$

Proof. Trivial.

□

Lemma 5.2. *Let* $H = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \mid a, d \in F^\times, c \in F \right\}$ *and* $L = \left\{ \begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix} \mid y \in F^\times, x \in F \right\}$. *Then* H *and* L *are subgroups of* $\mathrm{GL}(2, F)$ *and we have*

$$M_{\psi_A} \simeq H \times L.$$

Proof. For $m \in M_{\psi_A}$ (as in Lemma 5.1), consider the map $\phi : M_{\psi_A} \to H \times L$ given by

$$\phi(m) = \left(\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}, \begin{bmatrix} 1 & ra^{-1} \\ 0 & sa^{-1} \end{bmatrix} \right).$$

Clearly, ϕ is an isomorphism of M_{ψ_A} onto $H \times L$. Indeed, for $m_1, m_2 \in M_{\psi_A}$, we have

$$\begin{aligned} \phi(m_1 m_2) &= \left(\begin{bmatrix} a_1 a_2 & 0 \\ c_1 a_2 + d_1 c_2 & d_1 d_2 \end{bmatrix}, \begin{bmatrix} 1 & r_2 a_2^{-1} + r_1 s_2 a_2^{-1} a_1^{-1} \\ 0 & s_1 s_2 a_2^{-1} a_1^{-1} \end{bmatrix} \right) \\ &= \phi(m_1)\phi(m_2). \end{aligned}$$

It follows that ϕ is a homomorphism. It is trivial to see that ϕ is injective. The result follows by observing that $|M_{\psi_A}| = |H \times L|$.

□

Lemma 5.3. $H \simeq F^\times \times \overline{L}$.

Proof. Let $h = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in H$. Clearly we have $h = z\ell$ for some $z \in Z$ and $\ell \in \overline{L}$. Indeed, we have

$$h = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ ca^{-1} & da^{-1} \end{bmatrix} = z\ell.$$

Since Z and $\overline{L}$ are normal in H, it follows that $H \simeq F^\times \times \overline{L}$.

□

Theorem 5.4 (Main theorem)**.** *Let* θ *be a regular character of* $F_4^\times$ *and* $\pi = \pi_\theta$ *be an irreducible cuspidal representation of* G. *Let* $\rho_1 = \theta|_{F^\times} \otimes \mathrm{ind}_{\bar{U}}^{\bar{L}} \mu^w$ *and* $\rho_2 = \mathrm{ind}_U^L \mu$. *Then*

$$\pi_{N,\psi_A} \simeq \rho_1 \otimes \rho_2$$

as M_{ψ_A} *modules.*

We prove Theorem 5.4 by showing that the character Θ_{N,ψ_A} of π_{N,ψ_A} is equal to the character χ_ρ of $\rho = \rho_1 \otimes \rho_2$ for all elements $m \in M_{\psi_A}$.

5.1. **Character calculation for ρ.** Let $\rho_1 = \theta|_{F^\times} \otimes \mathrm{ind}_{\bar{U}}^{\bar{L}} \mu^w$ and $\rho_2 = \mathrm{ind}_U^L \mu$. In this section, we calculate the character of the representation $\rho = \rho_1 \otimes \rho_2$.

Lemma 5.5. *Let* μ *be a fixed non-trivial character of* U. *Consider the representation*

$$\rho_1 = \theta|_{F^\times} \otimes \mathrm{ind}_{\bar{U}}^{\bar{L}} \mu^w$$

of H. *Let* χ_{ρ_1} *be the character of* ρ_1. *We have*

$$\chi_{\rho_1}\left(\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \right) = \begin{cases} 0, & \text{if } a \neq d \\ \theta(a) \displaystyle\sum_{y \in F^\times} \mu\left(\begin{bmatrix} 1 & yca^{-1} \\ 0 & 1 \end{bmatrix} \right), & \text{if } a = d. \end{cases}$$

Proof. Let $t = \begin{bmatrix} 1 & 0 \\ x & y \end{bmatrix} \in \overline{L}$ and $\ell = \begin{bmatrix} 1 & 0 \\ ca^{-1} & da^{-1} \end{bmatrix}$. We have,

$$t\ell t^{-1} = \begin{bmatrix} 1 & 0 \\ x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ ca^{-1} & da^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -xy^{-1} & y^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x + yca^{-1} - xda^{-1} & da^{-1} \end{bmatrix}.$$

Since $\rho_1 = \theta|_{F^\times} \otimes \operatorname{ind}_{\bar{U}}^{\bar{L}} \mu^w$, using the character formula for the induced representation, we have

$$\begin{aligned}
\chi_{\rho_1}\left(\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}\right) &= \chi_{\rho_1}(z\ell) \\
&= \theta|_{F^\times}\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) \chi_{\operatorname{ind}_{\bar{U}}^{\bar{L}}(\mu^w)}\left(\begin{bmatrix} 1 & 0 \\ ca^{-1} & da^{-1} \end{bmatrix}\right) \\
&= \frac{\theta(a)}{|\overline{U}|} \sum_{\substack{t \in \overline{L} \\ t\ell t^{-1} \in \overline{U}}} \mu^w(t\ell t^{-1}) \\
&= \begin{cases} 0, \text{ if } a \neq d \\ \theta(a) \displaystyle\sum_{y \in F^\times} \mu\left(\begin{bmatrix} 1 & yca^{-1} \\ 0 & 1 \end{bmatrix}\right), \text{ if } a = d. \end{cases}
\end{aligned}$$

□

Lemma 5.6. *Let μ be a fixed non-trivial character of U. Consider the representation*

$$\rho_2 = \operatorname{ind}_U^L \mu$$

of H. Let χ_{ρ_2} be the character of ρ_2. We have

$$\chi_{\rho_2}\left(\begin{bmatrix} 1 & ra^{-1} \\ 0 & sa^{-1} \end{bmatrix}\right) = \begin{cases} 0, \text{ if } a \neq s \\ \displaystyle\sum_{y \in F^\times} \mu\left(\begin{bmatrix} 1 & ra^{-1}y^{-1} \\ 0 & 1 \end{bmatrix}\right), \text{ if } a = s. \end{cases}$$

Proof. Let $t = \begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix} \in L$ and $k = \begin{bmatrix} 1 & ra^{-1} \\ 0 & sa^{-1} \end{bmatrix}$. We have,

$$tkt^{-1} = \begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix} \begin{bmatrix} 1 & ra^{-1} \\ 0 & sa^{-1} \end{bmatrix} \begin{bmatrix} 1 & -xy^{-1} \\ 0 & y^{-1} \end{bmatrix} = \begin{bmatrix} 1 & -xy^{-1} + ra^{-1}y^{-1} + xsy^{-1}a^{-1} \\ 0 & sa^{-1} \end{bmatrix}.$$

Since $\rho_2 = \operatorname{ind}_U^L \mu$, using the character formula for the induced representation, we have

$$\begin{aligned}
\chi_{\rho_2}\left(\begin{bmatrix} 1 & a^{-1}r \\ 0 & a^{-1}s \end{bmatrix}\right) &= \chi_{\rho_2}(k) \\
&= \frac{1}{|U|} \sum_{\substack{t \in L \\ tkt^{-1} \in U}} \mu(tkt^{-1}) \\
&= \begin{cases} 0, \text{ if } a \neq s \\ \displaystyle\sum_{y \in F^\times} \mu\left(\begin{bmatrix} 1 & ra^{-1}y^{-1} \\ 0 & 1 \end{bmatrix}\right), \text{ if } a = s. \end{cases}
\end{aligned}$$

□

Theorem 5.7. *Let $\rho = \rho_1 \otimes \rho_2$ and χ_ρ be the character of ρ. For $m \in M_{\psi_A}$, we have*

$$\chi_\rho(m) = \begin{cases} \theta(a) & \text{if } c \neq 0, \text{ and } r \neq 0 \\ \theta(a)(q-1)^2 & \text{if } c = 0, \text{ and } r = 0 \\ -\theta(a)(q-1) & \text{if } c \neq 0, r = 0 \text{ or } c = 0, r \neq 0 \end{cases}$$

if $a = d = s$. Otherwise, $\chi_\rho(m) = 0$.

Proof. For $m \in M_{\psi_A} \simeq H \times L$, we see that

$$\chi_{\rho_1}\left(\begin{bmatrix} a & 0 \\ c & a \end{bmatrix}\right) = \begin{cases} \theta(a)(q-1), & \text{if } c = 0 \\ -\theta(a), & \text{if } c \neq 0 \end{cases}$$

and

$$\chi_{\rho_2}\left(\begin{bmatrix} 1 & ra^{-1} \\ 0 & 1 \end{bmatrix}\right) = \begin{cases} (q-1), & \text{if } r = 0 \\ -1, & \text{if } r \neq 0 \end{cases}.$$

Since $\chi_\rho = \chi_{\rho_1}\chi_{\rho_2}$, the result follows.

□

5.2. **Character calculation for π_{N,ψ_A}.**

Let

$$\mathrm{M}(2,2,r,q) = \{X \in \mathrm{M}(2,F) \mid \mathrm{Rank}(X) = r\},$$

$$S(r,\alpha,\beta) = \{X \in \mathrm{M}(2,2,r,q) \mid \mathrm{Tr}(X) = \alpha, \mathrm{Tr}(A{L_1}^{-1}X) = \beta\}.$$

For $m \in M_{\psi_A}$ we write

$$m = \begin{bmatrix} L_1 & \\ & L_2 \end{bmatrix} \text{ and } z = \begin{bmatrix} L_1 & X \\ & L_2 \end{bmatrix},$$

where $L_1 = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$, $L_2 = \begin{bmatrix} a & r \\ 0 & s \end{bmatrix}$ and $X \in \mathrm{M}(2,F)$.

Theorem 5.8. *Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of* $\mathrm{GL}(4,F)$ *and Θ_θ be its character. For $m \in M_{\psi_A}$, if $a \neq d$ or $a \neq s$, then*

$$\Theta_{N,\psi_A}(m) = 0.$$

Proof. We have

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^4} \sum_{X \in \mathrm{M}(2,F)} \Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)}.$$

Let $f(\lambda)$ be the characteristic polynomial of z. It is clear that

$$f(\lambda) = (\lambda - a)^2(\lambda - d)(\lambda - s).$$

If $a \neq d$ or $a \neq s$, then $f(\lambda)$ is clearly not a power of an irreducible polynomial over F. It follows from Theorem 2.1 that $\Theta_\theta(z) = 0$ and hence the result.

□

Theorem 5.9. *Let $m = \begin{bmatrix} L_1 & \\ & L_2 \end{bmatrix}$ where $L_1 = \begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$ and $L_2 = \begin{bmatrix} a & r \\ 0 & a \end{bmatrix}$. Suppose $c \neq 0$ and $r \neq 0$. Then, we have*

$$\Theta_{N,\psi_A}(m) = \theta(a).$$

Proof. It is easy to see that

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^4} \sum_{X \in \mathrm{M}(2,F)} \Theta_\theta \begin{bmatrix} L_1 & X \\ 0 & L_2 \end{bmatrix} \overline{\psi_A({L_1}^{-1}X)}$$

To calculate the character value, we write

$$\Theta_{N,\psi_A} = \frac{1}{q^4}\{K_0 + K_1 + K_2\}$$

according to the rank of the matrix X and compute each of these terms. We summarize the computations for K_1 and K_2 in the following tables.

Table 3. Computation for K_1

Partition of $\mathrm{M}(2,2,1,q)$	X	$\Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)}$	$\#S(1,\alpha,\beta)$
$X \in S(1,0,0)$	$\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}$	$(-1)^3\theta(a)(1-q)$	$(2q-2)$
$X \in S(1,\alpha,0)$, $\alpha \in F^\times$	$\begin{bmatrix} 0 & y \\ z & \alpha \end{bmatrix}$	$(-1)^3\theta(a)(1-q)$	$(2q-1)$
$X \in S(1,\alpha,\beta)$, $\alpha \in F, \beta \in F^\times$, $\alpha \neq a\beta$	$\begin{bmatrix} a\beta & -z^{-1}(\alpha - a\beta)a\beta \\ z & \alpha - a\beta \end{bmatrix}$	$(-1)^3\theta(a)\overline{\psi_0(\beta)}$	$(q-1)$
$X \in S(1,\alpha,\beta)$ $\alpha, \beta \in F^\times$, $\alpha = a\beta$	$\begin{bmatrix} a\beta & y \\ z & 0 \end{bmatrix}$	$(-1)^3\theta(a)\overline{\psi_0(\beta)}$	$(2q-1)$

Table 4. Computation for K_2

Partition of $\mathrm{M}(2,2,2,q)$	X	$\Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)}$	$\#S(2,\alpha,\beta)$
$X \in S(2,0,0)$	$\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}$	$(-1)^3\theta(a)(1-q)$	$(q-1)^2$
$X \in S(2,\alpha,0)$, $\alpha \in F^\times$	$\begin{bmatrix} 0 & y \\ z & \alpha \end{bmatrix}$	$(-1)^3\theta(a)(1-q)$	$(q-1)^2$
$X \in S(2,\alpha,\beta)$, $\alpha \in F, \beta \in F^\times$, $\alpha \neq a\beta$	$\begin{bmatrix} a\beta & y \\ z & \alpha - a\beta \end{bmatrix}$	$(-1)^3\theta(a)\overline{\psi_0(\beta)}$	q^2-q+1
$X \in S(2,\alpha,\beta)$, $\alpha,\beta \in F^\times$, $\alpha = a\beta$	$\begin{bmatrix} a\beta & y \\ z & 0 \end{bmatrix}$	$(-1)^3\theta(a)\overline{\psi_0(\beta)}$	$(q-1)^2$

For simplicity, we let $\Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)} = D_X$. A simple computation shows that we have

$$K_1 = \sum_{X \in \mathrm{M}(2,2,1,q)} \Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)} = A_1 + A_2 + A_3 + A_4$$

where we have

a) $A_1 = \sum_{X \in S(1,0,0)} D_X$
b) $A_2 = \sum_{\substack{X \in S(1,\alpha,0) \\ \alpha \in F^\times}} D_X$
c) $A_3 = \sum_{\substack{X \in S(1,\alpha,\beta) \\ \alpha \in F, \beta \in F^\times \\ \alpha \neq a\beta}} D_X$
d) $A_4 = \sum_{\substack{X \in S(1,a\beta,\beta) \\ \beta \in F^\times}} D_X$

Using Table 3, and computing A_1, A_2, A_3 and A_4, we have

a) $A_1 = (-1)^3\theta(a)(1-q)(2q-2)$
b) $A_2 = (-1)^3\theta(a)(1-q)(2q-1)(q-1)$
c) $A_3 = (-1)^3\theta(a)(q-1)(q-1)(-1)$
d) $A_4 = (-1)^3\theta(a)(2q-1)(-1)$.

It follows that

$$K_1 = \sum_{X \in \mathrm{M}(2,2,1,q)} \Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)} = \theta(a)(2q^3 - 2q^2 + 1). \tag{5.1}$$

Using Table 4, and doing similar calculations we see that

$$K_2 = \sum_{X \in \mathrm{M}(2,2,2,q)} \Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)} = \theta(a)(q^4 - 2q^3 + 2q^2 - q). \tag{5.2}$$

Trivially, we have

$$K_0 = \sum_{X \in \mathrm{M}(2,2,0,q)} \Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)} = \theta(a)(q-1). \tag{5.3}$$

From (5.1), (5.2) and (5.3), it follows that

$$\Theta_{N,\psi_A}(m) = \theta(a).$$

□

Theorem 5.10. *Let* $m = \begin{bmatrix} L_1 & \\ & L_2 \end{bmatrix}$ *where* $L_1 = \begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$ *and* $L_2 = \begin{bmatrix} a & r \\ 0 & a \end{bmatrix}$. *Suppose* $c \neq 0$ *and* $r = 0$. *Then, we have*

$$\Theta_{N,\psi_A}(m) = -\theta(a)(q-1).$$

Proof. Proceeding in a similar way as in Theorem 5.9, we can compute the character value. We record the calculations that we need in the following tables.

TABLE 5. Computation for K_1

Partition of $\mathrm{M}(2,2,1,q)$	X	$\Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)}$	$\#S(1,\alpha,\beta)$
$X \in S(1,0,0)$	$\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}$	If $y \neq 0$, $(-1)^3\theta(a)(1-q)$; If $y = 0$, $(-1)^3\theta(a)(1-q)(1-q^2)$	$(q-1)$; $(q-1)$
$X \in S(1,\alpha,0)$, $\alpha \in F^\times$	$\begin{bmatrix} 0 & y \\ z & \alpha \end{bmatrix}$	If $y \neq 0$, $(-1)^3\theta(a)(1-q)$; If $y = 0$, $(-1)^3\theta(a)(1-q)(1-q^2)$	$(q-1)$; q
$X \in S(1,\alpha,\beta)$, $\alpha \in F, \beta \in F^\times$, $\alpha \neq a\beta$	$\begin{bmatrix} a\beta & -z^{-1}(\alpha - a\beta)a\beta \\ z & \alpha - a\beta \end{bmatrix}$	$(-1)^3\theta(a)(1-q)\overline{\psi_0(\beta)}$	$(q-1)$
$X \in S(1,\alpha,\beta)$, $\alpha, \beta \in F^\times$, $\alpha = a\beta$	$\begin{bmatrix} a\beta & y \\ z & 0 \end{bmatrix}$	$(-1)^3\theta(a)(1-q)\overline{\psi_0(\beta)}$	$(2q-1)$

Table 6. Computation for K_2

Partition of $\mathrm{M}(2,2,2,q)$	X	$\Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)}$	$\#S(2,\alpha,\beta)$
$X \in S(2,0,0)$	$\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}$	$(-1)^3\theta(a)(1-q)$	$(q-1)^2$
$X \in S(2,\alpha,0)$, $\alpha \in F^\times$	$\begin{bmatrix} 0 & y \\ z & \alpha \end{bmatrix}$	$(-1)^3\theta(a)(1-q)$	$(q-1)^2$
$X \in S(2,\alpha,\beta)$, $\alpha \in F, \beta \in F^\times$, $\alpha \neq a\beta$	$\begin{bmatrix} a\beta & y \\ z & \alpha - a\beta \end{bmatrix}$	$(-1)^3\theta(a)(1-q)\overline{\psi_0(\beta)}$	q^2-q+1
$X \in S(2,\alpha,\beta)$, $\alpha,\beta \in F^\times$, $\alpha = a\beta$	$\begin{bmatrix} a\beta & y \\ z & 0 \end{bmatrix}$	$(-1)^3\theta(a)(1-q)\overline{\psi_0(\beta)}$	$(q-1)^2$

Using Table 5 and Table 6 and proceeding as in Theorem 5.9, we have

$$K_1 = \sum_{X \in \mathrm{M}(2,2,1,q)} \Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)} = \theta(a)(q-1)(-q^4+2q^2-q-1) \tag{5.4}$$

and

$$K_2 = \sum_{X \in \mathrm{M}(2,2,2,q)} \Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)} = \theta(a)(q-1)(q-q^2). \tag{5.5}$$

Trivially, we have

$$K_0 = \sum_{X \in \mathrm{M}(2,2,0,q)} \Theta_\theta(z)\overline{\psi_A({L_1}^{-1}X)} = \theta(a)(q-1)(1-q^2). \tag{5.6}$$

Combining 5.4, 5.5 and 5.6, we conclude that

$$\Theta_{N,\psi_A}(m) = -\theta(a)(q-1).$$

□

Theorem 5.11. *Let* $m = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}$ *where* $L_1 = \begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$ *and* $L_2 = \begin{bmatrix} a & r \\ 0 & a \end{bmatrix}$. *Then for* $r \neq 0$ *and* $c = 0$, *we have*

$$\Theta_{N,\psi_A}(m) = -\theta(a)(q-1).$$

Proof. The proof is similar to Theorem 5.10. □

Theorem 5.12. *Let* $m = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}$ *where* $L_1 = \begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$ *and* $L_2 = \begin{bmatrix} a & r \\ 0 & a \end{bmatrix}$. *Then for* $c = 0$ *and* $r = 0$, *we have*

$$\Theta_{N,\psi_A}(m) = \theta(a)(q-1)^2.$$

Proof. The result follows by using the multiplicative Jordan decomposition and the dimension calculation in Theorem 4.2. □

5.2.1. *Proof of the Main Theorem.* Summarizing the results of Section 5 (Theorem $5.7 - 5.12$), we see that

$$\Theta_{N,\psi_A}(m) = 0, \text{ if } a \neq d \text{ or } a \neq s$$

and

$$\Theta_{N,\psi_A}(m) = \begin{cases} \theta(a) & \text{if } c \neq 0, \text{ and } r \neq 0 \\ \theta(a)(q-1)^2 & \text{if } c = 0, \text{ and } r = 0 \\ -\theta(a)(q-1) & \text{if } c \neq 0, r = 0 \text{ or } c = 0, r \neq 0 \end{cases}$$

if $a = d = s$. Since

$$\Theta_{N,\psi_A}(m) = \chi_\rho(m), \forall m \in M_{\psi_A},$$

the result follows.

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References

1. S. D. Fisher and M. N. Alexander, *Classroom Notes: Matrices over a Finite Field*, Amer. Math. Monthly **73** (1966), no. 6, 639–641. MR 1533848
2. I. M. Gelfand and M. I. Graev, *Construction of irreducible representations of simple algebraic groups over a finite field*, Dokl. Akad. Nauk SSSR **147** (1962), 529–532. MR 0148765
3. S. I. Gelfand, *Representations of the general linear group over a finite field*, Lie groups and their representations (Proc. Summer School on Group Representations of the Bolya: János Math. Soc., Budapest, 1971) (1975), 119–132. MR 0442102
4. Ofir Gorodetsky and Zahi Hazan, *On certain degenerate Whittaker models for cuspidal representations of* $\mathrm{GL}_{k\cdot n}(\mathbb{F}_q)$, Math. Z. **291** (2019), no. 1-2, 609–633. MR 3936084
5. J. A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. **80** (1955), 402–447. MR 72878
6. N. Kawanaka, *Generalized gelfand-graev representations and ennola duality*, Algebraic groups and related topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math., vol. 6, North-Holland, Amsterdam, 1985, pp. 175–206. MR 803335
7. Dipendra Prasad, *The space of degenerate Whittaker models for general linear groups over a finite field*, Internat. Math. Res. Notices (2000), no. 11, 579–595. MR 1763857
8. Jean-Pierre Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, Vol. 42, Springer-Verlag, New York-Heidelberg, 1977, Translated from the second French edition by Leonard L. Scott. MR 0450380

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